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Albert-Ludwigs-Universität Freiburg
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Topology
Problem Sheet 4
Deadline: 14 May 2024, 15h

Exercise 1 (2 Punkte)**.**
Let Y be a subset of a topological space X equipped with the subspace topology. For any subset A of Y show that $\overset{\circ}{A} \cap Y$ is contained in the interior of A as a subset of Y, where $\overset{\circ}{A}$ is the interior of A in X.

Are the interior of A as a subset of Y and $\overset{\circ}{A} \cap Y$ always equal?

Exercise 2 (4 Points)**.**

a) Given a homeomorphism $f : X \to Y$ between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ as well as a subset $A \subset X$, show that the restriction map $f_{\restriction A} : A \to f(A)$ is again a homeomorphism, where A (resp. $f(A)$) is equipped with the subspace topology from X (resp. from Y).

b) Deduce from the above that there is no homeomorphism $f : [0,1] \to (0,1)$, both equipped with the subspace topology from the euclidean line.

c) Are $\mathbb{R}$ and $\mathbb{R}^2$ (both with the euclidean topology) homeomorphic?

Exercise 3 (8 Points)**.**
Fix some prime number p.

a) Given $m \leq n$ in $\mathbb{N}$, show that the map

$$\begin{array}{rccc} \phi_{m,n} : & \mathbb{Z}/p^n\mathbb{Z} & \to & \mathbb{Z}/p^m\mathbb{Z} \\ & x \mod p^n & \mapsto & x \mod p^m \end{array}$$

is well-defined.

For every n in $\mathbb{N}$ we consider the set $\mathbb{Z}/p^n\mathbb{Z}$ as a topological space with the discrete topology. Set now $X = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ with the product topology.

b) Show that the set $\mathbb{Z}_p = \{(x_n)_{n\in\mathbb{N}} \in X \mid \phi_{m,n}(x_n) = x_m \text{ for all } m \leq n \text{ in } \mathbb{N}\}$ is closed in X.

c) Giving some n_0 in $\mathbb{N}$, show that the set

$$\{(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p \mid x_n = 0 \text{ for all } n \leq n_0\}$$

is a clopen subset of $\mathbb{Z}_p$ with the subspace topology of X.

d) Show that $\mathbb{Z}_p$ is totally disconnected.

(Please turn over the page!)

Exercise 4 (6 Points)**.**

Consider the subset $A_0 = [0, 1] \times \{0\}$ of $\mathbb{R}^2$ and set

$$A_n = \{(x, y) \in \mathbb{R}^2 \mid x - ny = 0, x \in [0, 1]\} \text{ for } n \geq 1 \text{ in } \mathbb{N}.$$

Equip the union X of all A_n's with the subspace topology from $\mathbb{R}^2$ with the euclidean topology.

a) Is X connected? Is X path-connected?

b) Determine for which n in $\mathbb{N}$ the subset $A_n \setminus \{(0, 0)\}$ is open in X.

c) Is X locally connected?